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A closer look at the Russian roulette problem: A re-examination of the nonlinearity of the prospect theory's decision weight π

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ABSTRACT

Utilizing the Russian roulette problem as an exemplar, Kahneman and Tversky (1979) developed a weighting function π to explain that the Allais Paradox arises because people behave so as to maximize overall value rather than expected utility (EU). Following the way that "overweighting of small probabilities" originated from the Russian roulette problem, this research measured individuals' willingness to pay (WTP) as well as their happiness for a reduction of the probability of death, and examined whether the observed figures were compatible with the nonlinearity of the weighting function. Data analysis revealed that the nonlinear properties estimated by straight measures differed from those derived from preferential choices [D. Kahneman, A. Tversky, Prospect theory: an analysis of decision under risk, Econometrica 47 (1979) 263–291] and formulated by [A. Tversky, D. Kahneman, Advances in prospect theory: Cumulative representation of uncertainty, Journal of Risk and Uncertainty 5 (1992) 297–323]. The controversies and questions to the proposed properties of the decision weight were discussed. An attempt was made to draw the research attention from which function was being maximized to whether people behave as if they were trying to maximize some generalized expectation.

1. Introduction

The psychophysical approach to decision-making can be traced to the work of Daniel Bernoulli [2], where it is believed that it is utility, not outcome, that is weighted by its probability to form an expectation-type decision criterion: $EU = \sum_{i=0}^{n} p_i u(x_i)$. However, experimental work on risky decision-making provides empirical evidence that people systematically violate the expected utility (EU) principle. The best known counter-example is the Allais Paradox [1], which calls into question EU's independence axiom, *i.e.* the postulate that if A $\succ$ B then (A, p; C, $1-p$) $\succ$ (B, p; C, $1-p$) for any non-zero p and C (Note that (A, p; C, $1-p$) here refers to a prospect that yields A with probability p and C with probability $1-p$).

A weighting function π was developed by prospect theory [5] to explain that the Allais Paradox [1] arises because people behave so as to maximize overall value rather than EU. It replaces the objective probabilities by which utility is assumed to be weighted in EU theory by subjective decision weights, and specifies that it is the overall prospect value that is being maximized. In proposing such a function, a Russian roulette problem was utilized as an exemplar by Kahneman and Tversky [5] to illustrate the nonlinearity of a decision weight π:

> Suppose you are compelled to play Russian roulette, but are given the opportunity to purchase the removal of one bullet from the loaded gun. Would you pay as much to reduce the number of bullets from four to three as you would to reduce

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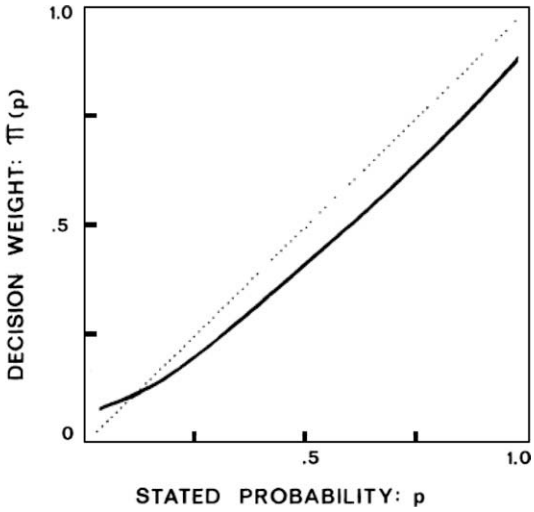

Fig. 1. A hypothetical weighting function adapted from Fig. 4, Kahneman and Tversky, 1979, p. 283 with permission.

> the number of bullets from one to zero? Most people feel that they would be willing to pay much more for a reduction of the probability of death from 1/6 to zero than for a reduction from 4/6 to 3/6. Economic considerations would lead one to pay more in the latter case, where the value of money is presumably reduced by the considerable probability that one will not live to enjoy it (Kahneman and Tversky, 1979, p. 283).

A decision weight π (see Fig. 1) was therefore derived from a number of preferential choices, where small probabilities are overweighted and medium to large probabilities are underweighted. These properties entail that π is relatively shallow in the open interval and changes abruptly near the end-points where $\pi(0) = 0$ and $\pi(1) = 1$.

The critical element of prospect theory [5] is the fact that probabilities, p, in decision-making situations are transformed nonlinearly into weights w(p). Tversky and Kahneman [21] used a one-parameter function, which transforms probabilities into weights and has the property of being regressive, asymmetric, and S-shaped:

$$w^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}$$
$$w^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}}$$

The one-parameter function is more flexible to account for various counterexamples that question the direct relationship between the 'experiments' and the derivations of decision weights [*e.g.*, 4,8,16,17]. On the other hand, this formula makes itself applicable and testable in the context of "reduction of the probability of death" and provides us with some guidance as to the route falsifying research might take.

Table 1 and Fig. 2 present the weighting change calculated using the above formula for a reduction of one bullet when the revolver was loaded with only 1 bullet (2, 3, 4, 5 and 6 bullets as well).

In studying the curvature of the probability weighting function, Wu and Gonzalez [22] also asserted that:

> Empirical studies have consistently suggested an inverse S-shaped weighting function, which is concave below and convex above some fixed $p^* < 0.40$... Returning to the Russian roulette example, concavity for small probabilities suggests that people would pay more to reduce the number of bullets from one to zero than from two to one, while convexity for large probabilities suggests a higher premium for a reduction from six to five than from five to four (Wu & Gonzalez, 1996, p. 1677).

Table 1
A reduction of the probability of death and the corresponding probability/weighting changed (ΔP/ΔW) given $\delta = 0.61$.

Number of bullet(s) loaded	Probability of death	$W(p)$	$\mathbf{\Delta P} = p - \frac{1}{6}$ (probability changed)	$W(p - \frac{1}{6})$	$\mathbf{\Delta W} = W(p) - W(p - \frac{1}{6})$ (weighting changed)
6	6/6	1	$\frac{1}{6}$	0.63728699	0.362713
5	5/6	0.63728699	$\frac{1}{6}$	0.512750077	0.1245369
4	4/6	0.512750077	$\frac{1}{6}$	0.420639	0.092116007
3	3/6	0.420639	$\frac{1}{6}$	0.3359521	0.09846869
2	2/6	0.3359521	$\frac{1}{6}$	0.23876098	0.09719112
1	1/6	0.23876098	$\frac{1}{6}$	0	0.23876098
0	0/6	0			

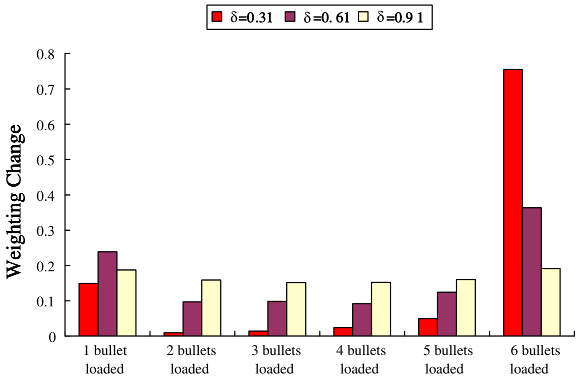

Fig. 2. $\Delta W = W(p) - W(p - \frac{1}{6})$ (weighting changed) given $\delta = 0.31$, 0.61 and 0.91 for a reduction of one bullet when the revolver loaded with 6, 5, 4, 3, 2, or only 1 bullet.

The above discussion implied a general agreement on the weighting function's inverse S-shaped curve, *i.e.* first concave and then convex. A U-shaped trend as illustrated in Fig. 2 will eventually be captured by this family of inverse S-shaped weighting functions.

The purpose of this research was to measure people's willingness to pay (WTP) as well as their happiness for a reduction of the probability of death, and to see whether the observed figures were compatible with the nonlinearity suggested by Tversky and Kahneman [21].

2. Method

2.1. Participants

The participants for the present research were 28 postgraduates from the Institute of Psychology, Chinese Academy of Sciences, and 42 undergraduates from Nankai University recruited through campus-wide advertisement. The undergraduates were paid ¥5 for participating while the postgraduates participated as volunteers.

2.2. Materials and procedure

The Russian roulette scenario was based on Kahneman and Tversky's [5] material and was presented on a PC screen. A betting task and a gripping task were prepared.

The betting task required participants to indicate the price (up to their own personal savings in cash) at which they would be willing to purchase the removal of one bullet from the loaded gun under conditions that the revolver loaded with 6, 5, 4, 3, 2, or only 1 bullet, respectively. The 28 postgraduates were assigned the task of betting only. In the gripping task, participants were asked to use an electron grip strength meter to reflect how happy they felt when one bullet was removed from the loaded gun, under conditions that the revolver was loaded with 6, 5, 4, 3, 2, or only 1 bullet, respectively. Both the betting and the gripping tasks were posed to the 42 undergraduates in a balanced order: half of them bet first and half gripped first.

3. Results

The results of our experiment are presented in Figs. 3, and 4. The WTP data and happiness data in terms of hand-grip strength coincided very well. A repeated ANOVA was conducted both on the betting and gripping data. The analysis revealed a significant main effect ($F(5, 345) = 5.76$, $p < .001$ for betting; $F(5, 205) = 4.89$, $p < .001$ for gripping), indicating that individuals would not be indifferent between each reduction of the same probability of death. The resulting main effect suggests that either the WTP or the strength of happiness for the removal of one bullet is not a constant at all.

A closer look at Figs. 3 and 4, however, indicated that these results were not compatible with the nonlinearity of the weighting function in that (a) either WTP or happiness for a reduction of one bullet was a decreasing monotonic function of the number of bullets loaded, and (b) neither WTP nor happiness for a reduction of one bullet changed abruptly near the unity-point. The data obtained suggested a nonlinear shape that was neither asymmetric nor U-shaped as shown in Fig. 2.

4. General discussion and conclusions

The apparent need for a fundamental change in expected utility (EU) theory is largely attributable to expected utility's linearity assumption, which has been demonstrated by the Allais Paradox that it is in contradiction to EU maximization.

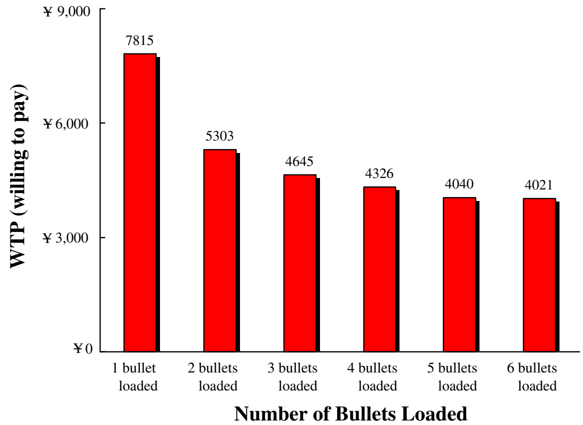

Fig. 3. WTP (willing to pay) in Chinese Yuan for a reduction of one bullet when the revolver loaded with 6, 5, 4, 3, 2, or only 1 bullet ($N = 70$). A curve-fitting formula, $y = e^{8.211+0.806/x}$, which describes the WTP data was suggested.

The Russian roulette problem discussed in this paper has provided grist for the theoretical mill in the sense that prospect theory [5] has been proposed in an attempt to resolve the paradox by assuming that the utilities of outcomes are not multiplied by linear probabilities, but by nonlinear decision weights $\pi(p)$. These decision weights express the subjective evaluation of objective probabilities. The parameter π is an increasing function of p but is not a probability, and, therefore, does not necessarily conform to the rules of mathematical probability. However, sharp-eyed readers will also have noticed that the nonlinearity of the decision weight $\pi(p)$ is not a straight measure of the subjective evaluation (*e.g.*, "willing to pay much more for a reduction of the probability of death from 1/6 to zero" presumably measured "overweighting of small probabilities"), but an artifact of deductive reasoning. That is, the properties of the decision weight $\pi(p)$ are derived by a deductive process which assumes that the option chosen by a gambler is the one that maximizes the overall worth of a gamble. The logic of postulating a nonlinear weighting function actually adds no evidence that the expectation principle itself is true (for detailed arguments see [11]). To appreciate the significance of this, note that the psychophysical principle that governs the perception of the horizon may have led to the impression that the Earth's surface is flat (somewhat analogously, expected utility is maximized); it does not constitute proof.

It would be interesting to see that Li [10] obtained redundant data disconfirming the weighting function by using the exact method employed by Kahneman and Tversky [5]. That is, the test was performed by assuming that the decision maker's preferential choice was the one which maximized something. The option chosen by over 50% of the participants was the one with the greatest overall value. The observed behavior, however, yielded self-contradictory properties of the weighting function π.

In order to re-examine the nonlinear properties in a less circuitous way, the present research investigated the nonlinearity by measuring people's nonlinear feeling for a reduction of the probability rather than by deriving it from preferential choices [5]. From these measures we draw two conclusions: First, the observed data is nonlinear, but the nonlinearity ob-

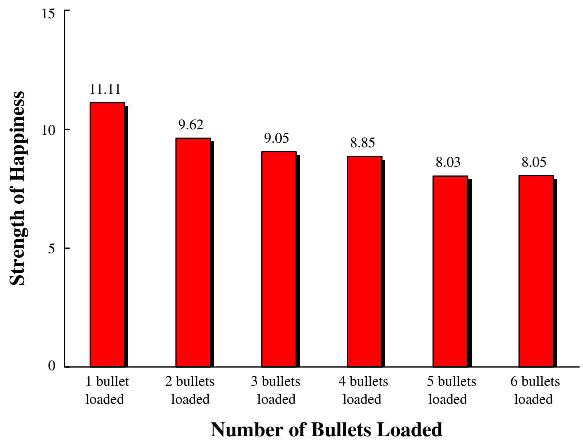

Fig. 4. The strength of happiness measured by electron grip strength meter in Newton for a reduction of one bullet when the revolver loaded with 6, 5, 4, 3, 2, or only 1 bullet ($N = 42$). A curve-fitting formula, $y = 11.057 \cdot x^{-0.182}$, which describes the happiness was suggested.

tained empirically does not fit the one predicted by the decision weight w [21]; Second, and more importantly, the decreasing monotonic function of the reduction of the probability cannot be absorbed by any S-shaped weighting function.

In decision science, the psychophysical approach to decision-making was first applied in the analysis of responses to money and then extended to the analysis of responses to probability. The psychophysics of values which introduced the concept of utility and the psychophysics of chance which introduced the concept of subjective probabilities were both heavily exploited in modifying the form of expectation $\sum x_i p_i$. If objective value or probability fails to apply in reality, then they should be replaced by subjective ones, but there is still an attempt to maximize some kind of expectation.

The discrepancy between measurements of utility or subjective probability in the presence and absence of risk has led us to question why people's actual preferential choice is, by default, seen as a process of expectation maximization, and why psychophysics presumably plays a weighted-role in such a process. If people's actual choice process on which the derivation is based is not an expectation maximization, but a satisfying [19], or an "admissibility" [7], or a priority heuristic [3], or an equate-to-differentiate [9,14], or other possible successful candidates, the alluded discrepancy will eventually exist in one way or another. For example, utility functions of money derived from risky choices differ from those derived from direct observation, *e.g.*, the Markowitz's type of utility [15], first concave and then convex, is different from the classic utility of money, concave in the positive region, reflecting the phenomenon of diminishing marginal utility. In the same vein, the direct measure of people's nonlinear feeling for the reduction of the probability in the present study contradicts the U-shaped relationship between the probability change and the subjective intensity of it determined by applying the weighting function $w(p)$.

This is more easily seen by way of an example illustrated as Tversky-intransitivity stimuli [20] that involves the following five lotteries: (a) (5.00, 7/24); (b) (4.75, 8/24); (c) (4.50, 9/24); (d) (4.25, 10/24); (e) (4.00, 11/24). The first four lotteries are constructed so that the probabilities of adjacent lotteries are considered to be identical and the choice between lotteries are all based on payoff (x) dimension, whereas the probabilities of (a) and (e) are designed to differ enough to affect evaluation and choice based on chance (p) dimension. However, a careful reader would wonder why participants' responses to the Tversky-intransitivity stimuli, which are constructed to make choices in line with a non-compensatory (non-expectation maximizing) model, are not being explained by Tversky so as to be elicited by a non-compensatory process (which assumes decision is based on only payoff (x) or chance (p) dimension) but by a compensatory one (which assumes that people choose according to an ordering implied by an aggregate of judgments of all dimensions) (for more detailed argument see [12,13]).

It is our conjecture that the deductive measurement of utility or subjective probability based on a non-expectation assumption (*e.g.,* similarity models [6,18]) should yield a different result as that based on an expectation assumption (*e.g.,* prospect theory). If and only if the assumed choice process on which derivation is based fit the actual one, the utility or subjective probability measured by using this deductive method should yield the same result as that measured in the riskless situation. In order to measure the utility or subjective probability in Tversky-intransitivity case [20], it should be assumed that the chosen option is NOT the one which has the greater expectation but the one which is better on the determinant dimension (payoff (x) or chance (p) dimension), *i.e.* the dimension with greater dimensional difference, where the diminishing sensitivity will lend support to determine which dimensional difference is the subjectively, but not objectively, greater one (for similar analysis in riskless cases, see [14]).

In sum, the present analysis does not attempt to present an alternative weighting function that might agree more with the present findings than does Kahneman and Tversky's. Rather, it suggests that Li's [10] results are not at all surprising if the expectation hypothesis is itself deemed to be false and attempts to draw the research attention from which function was being maximized to whether people behave as if they are trying to maximize some generalized expectation.

It is also suggested that the use of psychophysical functions in decision-making under risk has been misguided since the time of Bernoulli.

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